MARSHALL'S THEORY OF VALUE AND THE STRONG LAW OF DEMAND

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Abstract

We show that all the fundamental properties of competitive equilibrium in Marshall's theory of value, as presented in Note XXI of the mathematical appendix to his Principles of Economics (1890), derive from the Strong Law of Demand. This is, existence, uniqueness, optimality, global stability of equilibrium prices with respect to tatonnement price adjustment and refutability follow from the cyclical monotonicity of the market demand function in the Marshallian general equilibrium model.

Keywords: Partial equilibrium analysis, short run equilibrium, strong law of demand, cyclical monotonicity, Legendre-Fenchel duality

JEL Classification: B13, C62, D11, D51

1 Introduction

Marshall in NOTE XXI of the mathematical appendix to his Principles of Economics (1890) presents a fully articulated theory of general equilibrium in market economies. His model differes in several fundamental ways from the general equilibrium model of Walras (1900). In Marshall's model there are no explicit budget constraints for consumers, the marginal utilities of incomes are exogenous constants and market prices are not normalized. He "proves" the existence of market clearing prices, as does Walras, by counting the number of equations and unknowns. Marshall's first order conditions for consumer satisfaction require the gradient of the consumer's utility function

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at the optimal consumption bundle to equal the vector of market prices, unlike the Walrasian model where the first order conditions for consumer satisfaction require the normalized gradient of the consumer's utility function at the optimal consumption bundle to equal the normalized vector of prices.

This is not the partial equilibrium model with only two goods usually associated with Cournot (1838), Dupuit (1844) or Marshall (1890), nor is is it the partial equilibrium model exposited in the first chapter of Arrow and Hahn (1971), or in chapter 10 of Mas-Colell, Whinston and Green (1995)(MWG). A recent modern discussion of the fundamental properties of Marhsall's general equilibrium model in NOTE XXI can be found in chapter 8 of Bewley (2007), where he calls it "short-run equilibrium". Consumers in Bewley's model satisfy Marshall's first order conditions in a short run equilibrium. Bewley proves that: (i) a unique short-run equilibrium exists, (ii) welfare in a short-run equilibrium can be computed using consumer surplus, and (iii) the short-run equilibrium is globally stable under tatonnment price adjustment.

For ease of exposition we limit most of our discussion to pure exchange models but, as suggested by the analysis of Marshall's theory of value in Bewley (2007), all of our results extend to Marshallian general equilibrium models with production. We summarize the general case after our discussion of pure exchange economies.

We prove that all properties shown in Bewley (2007) of Marshall's general equilibrium model, are an immediate consequence of the market demand function satisfying the Strong Law of Demand. In Brown and Calsamiglia (2007) a demand function is said to satisfy the Strong Law of Demand if it is a cyclically monotone function of prices. Cyclically monotone demand functions do not only have downward sloping demand curves, in the sense that they are monotone functions, but also their line integrals are path-independent and measure the change in consumer's welfare in terms of consumer's surplus for a given multidimensional change in market prices. Following Quah (2000), we show that the Strong Law of Demand is preserved under aggregation across consumers.

Brown and Calsamiglia (2007) prove that if a demand function satisfies the Strong Law of Demand, then the consumer behaves as if he was maximizing a quasi-linear utility function subject to a budget constraint. The assumption on quasi-linearity of utility functions is made by MWG in their discussion of partial equilibrium analysis in the two good case, but there is no explicit mention of quasi-linear utilities or the Strong Law od Demand in Bewley's discussion of short-run equilibrium. Both Bewley and MWG partial and short-run equilibrium analysis derive from cyclical monotonicity of the market demand function in Marshall's general equilibrium model.

Using the Legendre-Fenchel duality theory for smooth concave functions presented in Rockafellar (1970), we offer a characterization of the Marshallian general equilibrium model as a representative agent model. This description is due to Bewley (1980) and allows us to prove that Marshall's theory of value is refutable. That is, there exists a finite family of multivariate polynomial inequalities, the Afriat inequalities for quasi-linear utilities derived by Brown and Calsamiglia (2007), where the parameters are the market prices and aggregate demands and the unknowns are the utility levels and marginal utilities of income of the representative consumer. They show that these inequalities have a solution if and only if the finite data set consisting of observations on market prices and associated aggreagate demands is cyclically monotone- see Theorem 1 in Brown and Calsamiglia (2007) and the first chapter of Brown and Kubler (2007) for a general discussion of refutable theories of value.

2 The Strong Law of Demand

Hildenbrand's (1983) extension of the law of demand to multicommodity market demand functions requires the demand function to be monotone. He showed that it is monotone if the income distribution is price independent and has downward sloping density. Subsequently, Quah (2000) extended Hildenbrand's analysis to individual's demand functions. His sufficient condition for monotone individual demand is in terms of the income elasticity of the marginal utility of income. Assuming that the commodity space is $\mathbb{R}^n_{++}$, we denote the demand function at prices $p \in \mathbb{R}^n_{++}$ by $x(p)$. This demand function satisfies the law of demand or is monotone if for any pair $p, p' \in \mathbb{R}^n_{++}$ of prices

$$(p - p') \cdot \left[x(p) - x(p')\right] < 0$$

This means, in particular, that the demand curve of any good is downward sloping with respect to its own price, i.e., satisfies the law of demand if all other prices are held constant.

In an unrelated exercise, Rockafellar (1970) introduced the notion of cyclical monotonicity as a means for characterizing the subgradient correspondance of a convex function. For smooth strictly convex functions f the gradient map $\partial f(x)$ is cylically monotone if for all finite sequences

$\{p_t, x_t\}_{t=1}^T$, where $p_t = \partial f(x_t)$:

$$x_1 \cdot (p_2 - p_1) + x_2 \cdot (p_3 - p_2) + \cdots + x_T \cdot (p_1 - p_T) \geq 0.$$

3 Marshall's General Equilibrium Model

We denote the Marshallian consumer optimization problem by (M):

$$\max_{x_i \in \mathbb{R}^n_{++}} \frac{1}{\lambda_i} g_i(x_i) - p \cdot x_i$$

where g_i is a smooth, strictly increasing and strictly concave utility function on $\mathbb{R}^n_{++}$, λ_i is the exogenous marginal utility of income, p is the vector of market prices and x_i is the consumption bundle. In this model there are no budget constraints and prices are not normalized. For a discussion of these assumptions in the context of rational expectations and Friedman's permanent income hypothesis, see chapter 8 in Bewley (2007). This specification of the consumer's optimization problem rationalizes the family of equations defining Marshall's general equilibrium model (absent production) in his NOTE XXI.

Let $x(p)$ be the solution to (M). (M) need not have a solution for all $p \in \mathbb{R}^n_{++}$, but as noted in Bewley (1980) the set of p such that (M) has a solution is nonempty, open and convex. Given his assumptions on g, it follows from Hadamard's Theorem — see Gordon (1972) for a discussion and proof of Hadamard's Theorem — that (M) has a solution for all $p \in \mathbb{R}^n_{++}$ if and only if the gradient map, i.e., $x \to \partial g(x)$, is a proper map. Recall that a continuous map $\ell : V \to W$ is proper if for every compact subset $K \subset W, \ell^{-1}(K)$ is a compact subset of V. In Marshall's specification of individual's utilities, where consumers have smooth additively separable utility functions, the marginal utility of consumption of each good goes to infinity as consumption goes to zero and the marginal utility of consumption of each good goes to zero as consumption goes to infinity, hence the gradient map is proper (e.g., $g(x) = \ln x$ has a proper gradient map, but $g(x) = x - e^{-x}$ does not). Marshall's demand functions are defined on all of $\mathbb{R}^n_{++}$. Bewley (1980) is therefore a generalization of Marshall, where Bewley drops Marshall's assumptions of separability of the utility function and properness of the gradient map. As in Marshall, we assume that $x \to \partial g(x)$ is a proper map, but we do not assume that utility functions are additively separable. In this case there is an equivalent formulation of (M) as a Walrasian consumer optimization problem, where the consumer maximizes a quasi-linear utility function subject to her budget constraint- see Brown and Calsamiglia (2007).

Theorem 1 *If there are I consumers, where each consumer i's optimization problem is given by (M), then the market demand function satisfies the Strong Law of Demand.*

Proof: Let $h_i(p) = \frac{1}{\lambda_i} g_i(x_i(p)) - p \cdot x_i(p)$ be the optimal value function for (M) for consumer i. Applying the envelope theorem we know that $\partial h_i(p) = -x_i(p)$.[1] Let $H(p) = \sum_{i=1}^{I} h_i(p)$, then $\partial H(p) = \sum_{i=1}^{I} \partial h_i(p) = \sum_{i=1}^{I} -x_i(p)$. Therefore the market demand at prices p is $X(p) = \sum_{i=1}^{I} x_i(p) = -\sum_{i=1}^{I} \partial h_i(p) = -\partial H(p)$.[2] Since $h_i(p)$ is a convex function,$-h_i(p)$ is concave and $-H(p)$ is also concave. Hence, the market demand function satisfies the Strong Law of Demand because the gradient map of a concave function is cyclically monotone–see Theorem 24.8 in Rockafellar (1970).

□

Corollary 1 The Marshallian general equilibrium model has a unique equilibrium price vector that is globally stable under tatonnment price adjustment.

Proof: Every cyclically monotone map is a monotone map. That is, market demand functions satisfying the Strong Law of Demand a fortiori satisfy the Law of Demand. Hildenbrand (1983) shows that economies satisfying the Law of Demand have a unique equilibrium price vectors that are globally stable under tatonnment price adjustment.

□

Corollary 2 Welfare in the Marshallian general equilibrium model can be computed using consumer surplus.

Proof: Brown and Brown (2007) show that this property of cyclically monotone demand functions.

□

[1] Convex analysis has a rich theory of duality, described by Rockafellar's (1970), based on the Legendre–Fenchel transform of a concave function $g(x)$, denoted $g^*(p)$, called the conjugate of $g(x) : g^*(p) = \inf_{x \in \mathbb{R}^n_{++}} \{p \cdot x - g(x)\}$. Hence $g^*(p) = -h(p)$ and h is an extended real-valued function. The conjugate (or surplus function in Bewley's terminology) plays the same role in analysis of the Marshallian consumer optimization model as the indirect utility function does in the Walrasian model of consumer choice.

[2] Theorem 16.4 in Rockafellar (1970) shows that the operations of addition and infimal convolution of proper convex functions are dual to eachother.

3.1 Marshallian Representative Consumer

To prove that the Marshallian general equilibrium model is refutable, we will first show that it can be described as a representative agent model, as originally suggested by Bewley (1980).

The representative agent's utility function in Bewley's Marshallian model is given by the following social welfare function:

$$W(e) = \max_{\{x_1,\ldots,x_I\}\in\mathbb{R}^{nI}_{++}} \left[\sum_{i=1}^{I}\frac{1}{\lambda_i}g_i(x_i)\right]$$
$$\text{s.t. } \sum_{i=1}^{I} x_i = e$$

Bewley shows that $(\bar{p}, x(\bar{p}))$ is an equilibrium of the exchange economy with consumers $\{(g_i,\lambda_i)\}_{i=1}^{I}$ and endowment $\bar{e}$ if and only if $\bar{e} = \text{argmax}\{W(e) - \bar{p}e\}$. Equivalently, for a given $\bar{e}$, the price vector $\bar{p}$ such that $\bar{e} = \text{argmax}\{W(e) - \bar{p}e\}$ will be the unique competitive equilibrium price vector for this exchange economy. Let $H(\bar{p}) = \max_{e\in\mathbb{R}^n_{++}}\{W(e) - \bar{p}\cdot e\}$, then it follows from this result that $H(\bar{p}) \equiv \sum_{t=1}^{T} h_t(\bar{p})$ if $\bar{p}$ is a competitive equilibrium vector of prices. Hence $-(\partial H/\partial p)|_{\bar{p}} = \sum_{t=1}^{T} -(\partial h_t/\partial p)|_{\bar{p}} = \sum_{t=1}^{T} x_t(\bar{p}) = x(\bar{p}) = \bar{e}$. The equilibrium map $p(e)$ is again the inverse of the demand function of the representative consumer. From Rockafellar (1970), Corollary 23.5.1, p. 219 we know that if g is a continuous concave function on $\mathbb{R}^n_{++}$ then $p \in \partial g(x)$ if and only if $x \in \partial - h(p)$. It follows from this duality relationship that $\bar{p}$ is the unique equilibrium price vector for the social endowment $\bar{e}$ if and only if $\bar{p} = (\partial W/\partial e)|_{e=\bar{e}}$ and $-(\partial H/\partial p)|_{\bar{p}} = \bar{e}$. Given a finite set of observations on social endowments and market clearing prices, we can now characterize the testable implications of Marshall's theory of value. A given data set rationalizes Marshall's general equilibrium model if and only if it is cyclically monotone.

Theorem 2 *The equilibrium map, $p(e)$, in Marshall's general equilibrium model is cyclically monotone in e, the social endowment.*

Proof: Because g_i is strictly concave, W is strictly concave as well. By Theorem 24.8 in Rockafellar (1970) we know that the gradient map of a concave function is cyclically monotone, which implies that the gradient map $\bar{e} \to (\frac{\partial W}{\partial e})|_{e=\bar{e}} = \bar{p}$ is cyclically monotone.

□

All of our results: existence, uniqueness, optimality, tatonnment stability and refutability extend to the Marshallian general equilibrium model with production. Optimality, tatonnment stability and refutability follow from the well-known duality result in convex analysis that the supply function is the gradient of the profit function or conjugate of the cost function. As such, the supply function is also cyclically monotone .The cyclical monotonicity of aggregate supply and aggregate demand guarantee (i) that producer and consumer surplus are well defined, (ii) that the excess demand function is cyclically monotone and (iii) that the aggregate demand function and the aggregate supply function are refutable .As in Bewley (2007), existence is shown by maximizing the representative agent ‘s utility function over the compact set of feasible production plans .If this set is strictly convex then the optimum is unique and the supporting prices are the equilibrium prices .See Bewley‘s chapter on short-run equilibria for detailed proofs of existence ,uniqueness ,optimality and tatonnment stability .Refutability follows from Brown and Calsamiglia (2007).

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